On integrability of one third-order nonlinear evolution equation

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Abstract

We study one third-order nonlinear evolution equation, recently introduced by Chou and Qu in a problem of plane curve motions, and find its transformation to the modified Korteweg–de Vries equation, its zero-curvature representation with an essential parameter, and its second-order recursion operator.

1 Introduction

In their recent work on the motions of plane curves [1], Chou and Qu found the following new third-order nonlinear evolution equation:

$$u_t = \frac{1}{2}\left((u_{xx}+u)^{-2}\right)_x . \tag{1}$$

"We do not know if this equation arises from the AKNS- or the WKI-scheme", wrote Chou and Qu in [1].

In the present paper, we study integrability of (1). In Section 2, we find a chain of Miura-type transformations, which relates the equation (1) with the modified Korteweg–de Vries equation (mKdV). In Section 3, using the obtained transformations and the well-known zero-curvature representation (ZCR) of the mKdV, we derive a complicated nontrivial ZCR of (1), which turns out to be neither AKNS- nor WKI-type one; and then we prove that simpler ZCRs of the equation (1) are trivial. In Section 4, we derive a second-order recursion operator of (1) from the obtained ZCR. Section 5 gives some concluding remarks.

2 Transformation to the mKdV

Let us try to transform the equation (1) into one of the well-known integrable equations. We do it, following the way described in [2]; further details on Miura-type transformations of scalar evolution equations can be found in [3].

First, we try to relate the equation (1) with an evolution equation of the form

$$v_t = v^3 v_{xxx} + g\left(v, v_x, v_{xx}\right) \tag{2}$$

by a transformation of the type

$$v\left(x,t\right) = a\left(u, u_x, \ldots, u_{x\ldots x}\right). \tag{3}$$

If there exists a transformation (3) between an evolution equation

$$u_t = f\left(u, u_x, u_{xx}, u_{xxx}\right): \quad \partial f / \partial u_{xxx} \neq \text{constant} \tag{4}$$

and an equation of the form (2), then necessarily

$$a = \left(\partial f / \partial u_{xxx}\right)^{1/3}. \tag{5}$$

Applying the transformation

$$\left(x, t, u\left(x,t\right)\right) \mapsto \left(x, t, v\left(x,t\right)\right): \quad v = -\left(u_{xx} + u\right)^{-1} \tag{6}$$

to the equation (1), we find that (6) really works and relates (1) with the equation

$$v_t = v^3 v_{xxx} + 3v^2 v_x v_{xx} + v^3 v_x. \tag{7}$$

Second, we notice that (7) can be written in the form

$$v_t = v^2 \left(v v_{xx} + v_x^2 + \tfrac{1}{2} v^2\right)_x. \tag{8}$$

Owing to this property, the equation (7) admits the transformation

$$\left(y, t, w\left(y,t\right)\right) \mapsto \left(x, t, v\left(x,t\right)\right): \quad x = w, \quad v = w_y, \tag{9}$$

which turns out to relate (7) with

$$w_t = w_{yyy} + \tfrac{1}{2} w_y^3. \tag{10}$$

And, third, we make the transformation

$$\left(y, t, w\left(y,t\right)\right) \mapsto \left(y, t, z\left(y,t\right)\right): \quad z = w_y \tag{11}$$

of (10) to the mKdV

$$z_t = z_{yyy} + \tfrac{3}{2} z^2 z_y, \tag{12}$$

for convenience in what follows, because $z\left(y,t\right) = v\left(x,t\right)$.

3 Zero-curvature representation

3.1 Transformation of the mKdV's ZCR

Using the chain of transformations (6), (9) and (11), we can derive a Lax pair for the equation (1), in the form of a ZCR containing an essential parameter, from the following well-known ZCR of the mKdV (12) [4, 5]:

$$\Phi_y = A\Phi, \quad \Phi_t = B\Phi, \quad D_t A = D_y B - [A, B] \tag{13}$$

with

$$A = \begin{pmatrix} \alpha & \frac{\mathrm{i}}{2}z \\ \frac{\mathrm{i}}{2}z & -\alpha \end{pmatrix}, \tag{14}$$

$$B = \begin{pmatrix} \frac{1}{2}\alpha z^2 + 4\alpha^3 & \frac{\mathrm{i}}{2}z_{yy} + \frac{\mathrm{i}}{4}z^3 + \mathrm{i}\alpha z_y + 2\mathrm{i}\alpha^2 z \\ \frac{\mathrm{i}}{2}z_{yy} + \frac{\mathrm{i}}{4}z^3 - \mathrm{i}\alpha z_y + 2\mathrm{i}\alpha^2 z & -\frac{1}{2}\alpha z^2 - 4\alpha^3 \end{pmatrix}, \tag{15}$$

where $\Phi(y,t)$ is a two-component column, D_t and D_y stand for the total derivatives, $[A, B]$ denotes the matrix commutator, and α is a parameter.

First of all, we obtain a ZCR for the equation (7) through the transformations (9) and (11). Introducing the column $\Psi : \Psi(x,t) = \Phi(y,t)$, we have $\Phi_y = z\Psi_x$, which allows to rewrite the equation $\Phi_y = A\Phi$ as

$$\Psi_x = X\Psi, \tag{16}$$

where $X = z^{-1}A$ after substitution of $z(y,t) = v(x,t)$,

$$X = \begin{pmatrix} \alpha v^{-1} & \frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & -\alpha v^{-1} \end{pmatrix}. \tag{17}$$

The equation $\Phi_t = B\Phi$, due to $\Phi_t = \Psi_t + w_t\Psi_x$ and $w_t = z_{yy} + \frac{1}{2}z^3$, leads to

$$\Psi_t = T\Psi, \tag{18}$$

where $T = B - \left(z^{-1}z_{yy} + \frac{1}{2}z^2\right)A$ after substitution of $z = v$, $z_y = vv_x$ and $z_{yy} = v^2 v_{xx} + v v_x^2$,

$$T = \begin{pmatrix} -\alpha\left(vv_{xx} + v_x^2\right) + 4\alpha^3 & \mathrm{i}\alpha v v_x + 2\mathrm{i}\alpha^2 v \\ -\mathrm{i}\alpha v v_x + 2\mathrm{i}\alpha^2 v & \alpha\left(vv_{xx} + v_x^2\right) - 4\alpha^3 \end{pmatrix}. \tag{19}$$

It is easy to check that the compatibility condition

$$D_t X = D_x T - [X, T] \tag{20}$$

of the equations (16) and (18), with the matrices X (17) and T (19), determines exactly the equation (7).

Then we can use the transformation (6). Substituting $v = -\left(u_{xx} + u\right)^{-1}$ into X (17) and T (19), we obtain a ZCR of the equation (1), in the sense that (20) is satisfied by any solution of (1). This ZCR, however, determines not the equation (1) itself, but a differential prolongation of (1),

$$u_{xxt} + u_t = \tfrac{1}{2}\left(\left(u_{xx} + u\right)^{-2}\right)_{xxx} + \tfrac{1}{2}\left(\left(u_{xx} + u\right)^{-2}\right)_{x}, \tag{21}$$

due to the structure of the transformed matrix X,

$$X = \begin{pmatrix} -\alpha\left(u_{xx} + u\right) & \frac{\mathrm{i}}{2} \\ \frac{\mathrm{i}}{2} & \alpha\left(u_{xx} + u\right) \end{pmatrix}. \tag{22}$$

The situation can be improved by a linear transformation of the auxiliary vector function Ψ,

$$\Psi \mapsto G\Psi, \quad \det G \neq 0, \tag{23}$$

which generates a gauge transformation of X and T,

$$X \mapsto GXG^{-1} + \left(D_x G\right) G^{-1}, \quad T \mapsto GTG^{-1} + \left(D_t G\right) G^{-1}. \tag{24}$$

The choice of

$$G = \begin{pmatrix} \exp\left(\alpha u_x\right) & 0 \\ 0 & \exp\left(-\alpha u_x\right) \end{pmatrix} \tag{25}$$

leads through (24) to the following gauge-transformed matrix X, which does not contain u_{xx}:

$$X = \begin{pmatrix} -\alpha u & \frac{\mathrm{i}}{2}\exp\left(2\alpha u_x\right) \\ \frac{\mathrm{i}}{2}\exp\left(-2\alpha u_x\right) & \alpha u \end{pmatrix}. \tag{26}$$

Note that u and u_x are separated in (26), and a ZCR with such a matrix X can determine an evolution equation exactly.

Now, from (19), (6), (24) and (25), we obtain the following matrix T, where u_t and u_{xt} have been expressed through (1) in terms of x-derivatives of u:

$$T = \begin{pmatrix} 4\alpha^3 & T_{12} \\ T_{21} & -4\alpha^3 \end{pmatrix} \tag{27}$$

with

$$T_{12} = -\mathrm{i}\alpha \exp\left(2\alpha u_x\right) \left(\frac{2\alpha}{u_{xx}+u} + \frac{u_{xxx}+u_x}{\left(u_{xx}+u\right)^3} \right), \tag{28}$$

$$T_{21} = \mathrm{i}\alpha \exp\left(-2\alpha u_x\right) \left(-\frac{2\alpha}{u_{xx}+u} + \frac{u_{xxx}+u_x}{\left(u_{xx}+u\right)^3} \right). \tag{29}$$

It is easy to check that the matrices X (26) and T (27)–(29) constitute a ZCR of (1), in the sense that the condition (20) with these matrices determines exactly the equation (1).

3.2 Simpler ZCRs are trivial

The obtained ZCR of (1) is characterized by the complicated matrix X (26) containing u_x. Does the equation (1) admit any simpler ZCR, with $X = X\left(x,t,u\right)$, of any dimension $n \times n$? This problem can be solved by direct analysis of the condition (20).

Substituting $X = X\left(x,t,u\right)$ and $T = T\left(x,t,u,u_x,u_{xx}\right)$ into (20) and replacing u_t by the right-hand side of (1), we obtain the following condition, which must be an identity, not an ordinary differential equation restricting solutions of (1):

$$X_t - \frac{u_{xxx}+u_x}{\left(u_{xx}+u\right)^3} X_u = D_x T - \left[X, T\right] \tag{30}$$

(here and below, subscripts denote derivatives, like $T_{u_x} = \partial T / \partial u_x$). Applying $\partial / \partial u_{xxx}$ and $\partial / \partial u_{xx}$ to the identity (30), we obtain, respectively,

$$T_{u_{xx}} = -\left(u_{xx}+u\right)^{-3} X_u, \tag{31}$$

$$T_{u_x} = \left(u_{xx}+u\right)^{-3} \left(D_x X_u - \left[X, X_u\right]\right). \tag{32}$$

The compatibility condition $\left(T_{u_{xx}}\right)_{u_x} = \left(T_{u_x}\right)_{u_{xx}}$ for (31) and (32) is $D_x X_u = \left[X, X_u\right]$, which is equivalent to

$$X = P\left(x,t\right) u + Q\left(x,t\right) : \quad P_x = \left[Q, P\right]. \tag{33}$$

Now, we make use of gauge transformations (24) with $G = G\left(x,t\right)$, choose G to be any solution with $\det G \neq 0$ of the system of ordinary differential equations $G_x = -GQ$, and thus set $Q = 0$ and $P = P\left(t\right)$ in the gauge-transformed matrix X (33). Then, $T_u = T_{u_{xx}}$ follows from $\partial / \partial u_x$ of (30), and this leads through the identity (30) to

$$T = \tfrac{1}{2} P\left(t\right) \left(u_{xx}+u\right)^{-2} + K\left(t\right) : \quad P_t = \left[K, P\right]. \tag{34}$$

Finally, we make $K = 0$ by a gauge transformation (24) with $G = G(t)$ satisfying $G_t = -GK$ and $\det G \neq 0$, and thus obtain

$$X = Pu, \quad T = \tfrac{1}{2} P \left(u_{xx} + u\right)^{-2}, \quad P = \text{constant}, \tag{35}$$

with any matrix P of any dimension $n \times n$. However, these matrices X and T (35) commute, $[X, T] = 0$, and the corresponding ZCR (20) is nothing but n^2 copies of the evident conservation law of the equation (1). In this sense, all the ZCRs sought, with $X = X(x, t, u)$, turn out to be trivial, up to gauge transformations (24) with arbitrary $G(x, t)$.

4 Recursion operator

Let us derive a recursion operator of the equation (1) from the matrix X (26) of its ZCR. We do it, following the way described in [6] (see also references therein). The recursion operator comes from the problem of finding the class of evolution equations

$$u_t = f\left(x, t, u, u_x, \ldots, u_{x \ldots x}\right) \tag{36}$$

that admit ZCRs (20) with the predetermined matrix X (26) and any 2×2 matrices $T\left(\alpha, x, t, u, u_x, \ldots, u_{x \ldots x}\right)$ of any order in $u_{x \ldots x}$.

The characteristic form of the ZCR (20) of an equation (36), with X given by (26), is

$$fC = \nabla_x S, \tag{37}$$

where C is the characteristic matrix,

$$C = \frac{\partial X}{\partial u} - \nabla_x \left(\frac{\partial X}{\partial u_x}\right), \tag{38}$$

the operator ∇_x is defined as $\nabla_x H = D_x H - [X, H]$ for any (here, 2×2) matrix H, and the matrix S is determined by

$$S = T - f \frac{\partial X}{\partial u_x}. \tag{39}$$

The explicit form of C (38) for X (26) is

$$C = \begin{pmatrix} 0 & -2\mathrm{i}\alpha^2 \mathrm{e}^{2\alpha u_x} \left(u_{xx} + u\right) \\ -2\mathrm{i}\alpha^2 \mathrm{e}^{-2\alpha u_x} \left(u_{xx} + u\right) & 0 \end{pmatrix}. \tag{40}$$

Under the gauge transformations (24) with any $G\left(\alpha, x, t, u, u_x, \ldots, u_{x \ldots x}\right)$, the characteristic matrix C is transformed as $C \mapsto GCG^{-1}$ [7], and therefore $\det C$ is a gauge invariant. We have $\det C = 4\alpha^4 \left(u_{xx} + u\right)^2$ in the case of (40), and this proves that the parameter α cannot be 'gauged out' from X (26), as well as that the matrix X (26) cannot be transformed by (24) into some X containing no derivatives of u.

Computing $\nabla_x C$, $\nabla_x^2 C$ and $\nabla_x^3 C$, we find the cyclic basis to be three-dimensional, $\{C, \nabla_x C, \nabla_x^2 C\}$, with the closure equation

$$\nabla_x^3 C = c_0 C + c_1 \nabla_x C + c_2 \nabla_x^2 C, \tag{41}$$

where

$$c_0 = \frac{u_{xxxxx} + 2u_{xxx} + u_x}{u_{xx} + u} - 9\frac{\left(u_{xxx} + u_x\right)\left(u_{xxxx} + u_{xx}\right)}{\left(u_{xx} + u\right)^2} + 12\frac{\left(u_{xxx} + u_x\right)^3}{\left(u_{xx} + u\right)^3}, \tag{42}$$

$$c_1 = 4\frac{u_{xxxx} + u_{xx}}{u_{xx} + u} - 12\frac{\left(u_{xxx} + u_x\right)^2}{\left(u_{xx} + u\right)^2} + 4\alpha^2 \left(u_{xx} + u\right)^2 - 1, \tag{43}$$

$$c_2 = 5\frac{u_{xxx} + u_x}{u_{xx} + u}. \tag{44}$$

Setting T to be traceless (without loss of generality), we decompose the matrix S (39) over the cyclic basis as

$$S = s_0 C + s_1 \nabla_x C + s_2 \nabla_x^2 C, \tag{45}$$

where s_0, s_1 and s_2 are unknown scalar functions of $x, t, u, u_x, \ldots, u_{x \ldots x}$ and α. Substitution of (45) into (37) leads us through (41) to

$$f = D_x s_0 + c_0 s_2, \quad s_0 = -D_x s_1 - c_1 s_2, \quad s_1 = -D_x s_2 - c_2 s_2, \tag{46}$$

where the function s_2 remains arbitrary. Then, from (46) and (42)–(44), we obtain

$$f = (M - \lambda N)\, r, \tag{47}$$

where $\lambda = 4\alpha^2$, $r\left(\lambda, x, t, u, u_x, \ldots, u_{x \ldots x}\right) = s_2$ is any function, of any order

in $u_{x\dots x}$, and the linear differential operators M and N are

$$\begin{aligned} M &= D_x^3 + 5\frac{u_{xxx}+u_x}{u_{xx}+u}D_x^2 \\ &\quad + \left(6\frac{u_{xxxx}+u_{xx}}{u_{xx}+u} + 2\frac{(u_{xxx}+u_x)^2}{(u_{xx}+u)^2} + 1\right)D_x \\ &\quad + \left(\frac{2u_{xxxxx}+3u_{xxx}+u_x}{u_{xx}+u}\right. \\ &\qquad \left. +4\frac{(u_{xxx}+u_x)(u_{xxxx}+u_{xx})}{(u_{xx}+u)^2} - 2\frac{(u_{xxx}+u_x)^3}{(u_{xx}+u)^3}\right), \end{aligned} \tag{48}$$

$$N = (u_{xx}+u)^2 D_x + 2(u_{xx}+u)(u_{xxx}+u_x). \tag{49}$$

Now, using the expansion

$$r = r_0 + \lambda r_1 + \lambda^2 r_2 + \lambda^3 r_3 + \cdots, \tag{50}$$

we obtain from (47) the expression for the right-hand side f of the represented equation (36), such that $\partial f/\partial\lambda = 0$ holds,

$$f = M r_0, \tag{51}$$

as well as the recursion relations for the coefficients $r_i(x, t, u, u_x, \dots, u_{x\dots x})$ of the expansion (50),

$$M r_{i+1} = N r_i, \quad i = 0, 1, 2, \dots. \tag{52}$$

The problem has been solved: for any set of functions $r_0, r_1, r_2, \dots$ satisfying the recursion relations (52), the expression (51) determines an evolution equation (36) admitting a ZCR (20) with the matrix X given by (26).

It only remains to notice that, if a set of functions $r_0, r_1, r_2, \dots$ satisfies the recursion relations (52), then the set of functions $r'_0, r'_1, r'_2, \dots$ determined by $r'_i = N^{-1} M r_i$ $(i = 0, 1, 2, \dots)$ satisfies (52) as well. Therefore the evolution equation $u_t = f'$ with $f' = M r'_0 = M N^{-1} M r_0 = M N^{-1} f = R f$ admits a ZCR (20) with X (26) whenever an equation $u_t = f$ does. Eventually, (48) and (49) lead us to the following explicit expression for the recursion operator $R = M N^{-1}$ of the equation (1):

$$R = \frac{1}{u_{xx}+u} D_x \frac{1}{u_{xx}+u}\left(D_x + D_x^{-1}\right). \tag{53}$$

5 Conclusion

Some remarks on the obtained results follow.

We succeeded in transforming the new Chou–Qu equation (1) into an integrable equation, the old and well-studied mKdV. The applicability of Miura-type transformations, however, is not restricted to integrable equations only. For instance, the original Miura transformation relates very wide (continual) classes of (mainly non-integrable) evolution equations [8].

We found the simplest nontrivial ZCR of the evolution equation (1). Its matrix X (26) contains u_x. For this reason, such a ZCR cannot be detected by those existent methods, which assume, as a starting point, that $X = X(x, t, u)$ must suffice in the case of evolution equations.

Of course, we could derive the obtained recursion operator (53) from the well-known recursion operator of the mKdV through the transformations found. However, we used a different method instead, mainly in order to illustrate, by this rather complicated example of $X = X(\alpha, u, u_x)$ (26), how the method works algorithmically.

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